A more general damping matrix can be decoupled if and only if

$$(i) \qquad [C][M]^{-1}[K] = [K][M]^{-1}[C]$$

and if

$$(ii) \qquad [\bar{C}_L] = [Y_L^T][C][X_L]$$

is not defective[3] in the real domain. $[X_L]$ and $[Y_L]$ are the right and the left eigenvectors corresponding to an L times repeated eigenvalue. It is obviously true of the necessity. To prove the sufficiency, suppose that the right and the left real mode matrices are denoted by $[X]$, $[Y]$, respectively. Hence,

$$[M] = [X][Y]^T \tag{20}$$

$$[\bar{K}] = [Y]^T[K][X] = \text{diag}(\alpha_1, \alpha_2, \ldots, \alpha_n) \tag{21}$$

Note $[\bar{C}] = [Y]^T[C][X]$, then condition (i) changes into $[\bar{C}][\bar{K}] = [\bar{K}][\bar{C}]$. Therefore, elements on both sides must be equal, i.e., $\bar{c}_{ij}\alpha_i = \bar{c}_{ij}\alpha_j$. If α_i is not equal to α_j, $\alpha_i \neq \alpha_j$, we will get $\bar{c}_{ij} = 0$ $(i \neq j)$. If α_i is an L times repeated eigenvalue, according to condition (ii) that $[\bar{C}_{ij}]$ is not defective in the real domain, there exist $[U]$, $[V]$ (dimensioned $L \times L$) to make

$$[V]^T[\bar{C}_L][U] = \text{diag}(\beta_1, \beta_2, \ldots, \beta_L) \tag{22}$$

Since $[X_L]$ and $[Y_L]$ are the eigenvectors corresponding to the L times repeated eigenvalue α_i, linear combinations $[X_L][U]$, $[Y_L][V]$ are also the eigenvectors of α_i. Consequently, $[C]$ can be decoupled in the real mode space.

A simple criterion to see whether $[\bar{C}]$ is defective or not in the real domain is that if $\{e_x\}^T\{e_y\} = 0$, then $[\bar{C}]$ is defective, where $\{e_x\}$ and $\{e_y\}$ are the left and the right eigenvectors of the same eigenvalue, respectively. If $\ll M, K \gg$ gives no repeated eigenvalues, $L = 1$; or if $\ll M, C, K \gg$ is of symmetry, which results in the symmetry of $[\bar{C}]$ and nondefectiveness in the real domain, then condition (ii) is naturally satisfied in these two cases. Thus, Caughey's condition becomes the necessary and sufficient one.

But if Eq. (17) is met with, the system $\ll M, C, K \gg$ is inevitable to be transformed into an equivalent undamped system described by Eq. (15), which is not the same as the decoupled one from the original system.

When $[C]$ can be decoupled, $[K_{eq}]$ can be simultaneously decoupled. In this case, Eq. (15) is reduced to

$$\{\ddot{p}\} + \left[\diagdown \; \omega_i^2 - \tfrac{1}{4}\bar{c}_{ii}^2 \; \diagdown\right]\{p\} = [Y]^T\{F(t)\} \tag{23}$$

V. Conclusions

On the basis of this analysis, a conclusion can be drawn that, if the Caughey's condition holds, a general $\ll M, C, K \gg$ system can be transformed into its equivalent undamped system so as to perform an analysis in the real domain instead of employing complex modal theories. Thus, the meaning of the Caughey's condition is broadened and its applicability is extended. In terms of decoupling general vibration equations, an extra condition is added in making such a determination.

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Nonlinear Free Vibration Characteristics of Laminated Anisotropic Thin Plates

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Introduction

LAMINATED composite plates and shell panels are being used in aerospace and other engineering applications as lightweight high-strength structural components. Analysis of free vibrations is one of the important design considerations for these components. The linear theory of vibrations predicts the frequency of natural vibrations to be independent of the amplitude. In many instances, if the amplitude of the vibration is large, such a result will not be justified due to one or another of the nonlinear effects. In general, the interest in vibrations of nonlinear systems centers on the vibrations of large amplitudes. Nonlinear vibrations of plates have received considerable attention in recent years. A comprehensive review of literature may be found in Refs. 1–4. The basic requirements for the analysis are the formulation of the equation of motion from the energy functions of the system and its solution for obtaining the vibration characteristics. The characteristic of the system, viz., the restoring force function in the equation of motion, is found to be a cubic polynomial, which is in the form of Duffing type or a combination of quadratic and cubic terms. The solution of the equation of motion (i.e., the nonlinear frequency of the plates that is a function of material properties, dimensions of the plates, and the amplitude of vibration) is obtained by several methods such as the perturbation method,[5–8] the harmonic balance method,[9] exact integration,[10–14] the iterative numerical schemes,[15,16] and the finite element method.[17] Some of the conventional tools for the analysis of nonlinear oscillations—such as the averaging techniques, multiple-time scaling, and harmonic balancing—are described in Ref. 18.

The usual perturbation techniques are inappropriate, if the coefficient of the nonlinear terms in the differential equation does not involve small parameters.[19] Mickens[20] has indicated that the only generally applicable technique in such a situation is the method of harmonic balance. Telban et al.[21] have recently demonstrated that the hybrid-Galerkin method improves the perturbation approximations of the Duffing equation. In recent years, Singh et al.[11] have shown that the perturbation solution suggested in Ref. 6 fails for the case of a simply supported two-layered antisymmetric cross-ply rectangular plate with immovable edges. The purpose of the present Note is to examine this case by solving the equation of motion through the hybrid-Galerkin method. The results are comparable with the exact solution of the equation of motion. The approximate solution of the equation of motion through the hybrid-Galerkin method discussed in this Note is quite simple and accurate and not only provides the frequency ratios for the specified amplitude ratios but also gives the expression for the displacement as a function of time.

Equation of Motion

The equation of motion for the nonlinear free vibration of a thin plate governed by the second-order nonlinear ordinary differential equation in time variable is of the form[5–16]

$$\frac{d^2W}{d\tau^2} + \omega_L^2 f(W) = 0 \tag{1}$$

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where the characteristic of the system $f(W)$ is

$$f(W) = W + \left(\frac{\beta}{\alpha}\right)W^2 + \left(\frac{\gamma}{\alpha}\right)W^3 \qquad (2)$$

where the linear frequency parameter is $\omega_L = (2\pi/T_L) = \sqrt{\alpha/\Sigma\rho_i t_i}$; T_L is the linear period; ρ_i and t_i are the density and thickness of the ith layer; the constants α, β, and γ are dependent on the material properties, dimensions of the plate, and wave numbers; and τ represents time.

The initial conditions corresponding to harmonic solutions of the period $2\pi/\omega$ are

$$W = W_{\max}, \qquad \frac{\mathrm{d}W}{\mathrm{d}\tau} = 0 \qquad \text{at } \tau = 0 \qquad (3)$$

where $W_{\max}$ is the maximum amplitude and ω the nonlinear frequency parameter.

Defining $\xi = W/t$ and $\phi = \omega\tau$ in Eqs. (1-3), one obtains

$$\left(\frac{\omega}{\omega_L}\right)^2 \ddot{\xi} + f(\xi) = 0 \qquad (4)$$

$$\xi = \xi_{\max}, \qquad \dot{\xi} = 0 \qquad \text{at } \phi = 0 \qquad (5)$$

where $f(\xi) = \xi + \delta_1\xi^2 + \delta_2\xi^3$, $\delta_1 = (\beta/\alpha)t$, $\delta_2 = (\gamma/\alpha)t^2$, $\xi_{\max} = (W_{\max}/t)$, $t = \Sigma t_i$ is the thickness of the plate, and overdots denote differentiation with respect to ϕ.

The hybrid-Galerkin method is used here for obtaining an approximate solution to the nonlinear differential equation (4). The approximate function $\bar{\xi}$ is chosen as

$$\bar{\xi} = \sum_{i=0}^{N} \xi_i \psi_i(\phi) = \sum_{i=0}^{N} \xi_i \cos(i\phi) \qquad (6)$$

which satisfies the homogeneous boundary conditions $\dot{\xi}(0) = \dot{\xi}(2\pi) = 0$ and has the period at $\tau = (2\pi/\omega)$, i.e., $\phi = 2\pi$.

From Eq. (4), the residual

$$\epsilon = \left(\frac{\omega}{\omega_L}\right)^2 \ddot{\bar{\xi}} + f(\bar{\xi}) \qquad (7)$$

is orthogonalized to each trial function ψ_i as

$$\langle \epsilon, \psi_i \rangle = \int_{\phi=0}^{\phi=2\pi} \epsilon\psi_i \,\mathrm{d}\phi = 0, \qquad i = 0, 1, 2, \ldots, N \qquad (8)$$

Both the approximate function $\bar{\xi}$ and the trial function ψ_i are substituted into Eq. (8), resulting in $N + 1$ equations that can be solved for the unknown coefficients ξ_i. For the specified amplitude ratio, $\xi = \xi_{\max}$, and assuming only two terms in Eq. (6), i.e., $N = 1$, the frequency ratio ω/ω_L is obtained by solving the following equations:

$$\xi_0 + \delta_1(\xi_0^2 + \tfrac{1}{2}\xi_1^2) + \delta_2(\xi_0^3 + \tfrac{3}{2}\xi_0\xi_1^2) = 0 \qquad (9)$$

$$\xi_1 + 2\delta_1\xi_0\xi_1 + 3\delta_2(\xi_0^2\xi_1 + \tfrac{1}{4}\xi_1^3) - \xi_1(\omega/\omega_L)^2 = 0 \qquad (10)$$

$$\xi_0 + \xi_1 - \xi_{\max} = 0 \qquad (11)$$

Equations (9) and (10) are obtained by using $\bar{\xi}$ and the trial functions in Eq. (8), whereas Eq. (11) is obtained by using Eq. (6) in Eq. (5). From Eqs. (9-11), the relationship between the frequency ratios ω/ω_L and the maximum amplitude ratio $\xi_{\max}$ is obtained as

$$\left(\frac{\omega}{\omega_L}\right)^2 = 1 + \frac{3}{4}\delta_2\xi_{\max}^2 + \left(2\delta_1 - \frac{3}{2}\delta_2\xi_{\max} + \frac{15}{4}\delta_2\xi_0\right)\xi_0 \qquad (12)$$

The value of ξ_0 for the specified $\xi_{\max}$ is obtained from the cubic equation

$$\sum_{k=0}^{3} a_k \xi_0^k = 0 \qquad (13)$$

where $a_0 = \tfrac{1}{2}\delta_1\xi_{\max}^2$, $a_1 = 1 - \delta_1\xi_{\max} + (3/2)\delta_2\xi_{\max}^2$, $a_2 = -3\delta_2\xi_{\max} + (3/2)\delta_1$, and $a_3 = (5/2)\delta_2$.

The perturbation solution of Chandra and Raju[6] given in Eq. (16) of Ref. 10 is written in the form

$$\left(\frac{\omega}{\omega_L}\right)^2 = 1 + \left(\frac{3}{4}\delta_2 - \frac{5}{6}\delta_1^2\right)\xi_{\max}^2 \qquad (14)$$

The arithmetic mean of the square of the upper and lower frequency ratios based on the simple harmonic oscillations assumption that is given in Eq. (24) of Ref. 10 is written in the form

$$\left(\frac{\omega}{\omega_L}\right)^2 = 1 + \frac{5}{6}\delta_1\xi_{\max} + \frac{3}{4}\delta_2\xi_{\max}^2 \qquad (15)$$

The perturbation solution given in Eq. (14) suggests that the sign of the amplitude does not affect the frequencies, whereas Eq. (15) indicates different values of frequencies. When $\delta_1 = 0$, the value of ξ_0 from the cubic equation for the specified $\xi_{\max}$ is found to be zero. It can be verified from Eqs. (12), (14), and (15) that these approximate solutions of the equation of motion for $\delta_1 = 0$ give the same result.

Example

To verify the adequacy of the hybrid-Galerkin method, a simply supported two-layered antisymmetric cross-ply rectangular plate with immovable edges for the first mode (i.e., the axial and transverse wave numbers, $m = n = 1$) examined in Ref. 11 is considered as an example problem. The dimensions of the plate are the following: length $a = 20$ mm, breadth $b = 10$ mm, and thickness $t = \Sigma t_i = 0.2$ mm (thickness of each layer $t_i = 0.1$ mm). The properties of the material considered are the following: $E_{11} = 20{,}000$ kg/mm², $E_{22} = 500$ kg/mm², $\nu = 0.25$, and $G_{12} = 250$ kg/mm². The constants α, β, and γ in

Table 1 Comparison of frequency ratios (ω/ω_L) for a simply supported two-layered cross-ply rectangular plate with immovable edges ($\delta_1 = -2.2518$ and $\delta_2 = 2.54328$)

					Present study	
$\xi_{\max}$	ξ_0, Eq. (13)	Perturbation method,[6] Eq. (14)	Simple harmonic oscillations,[10] Eq. (15)	Exact integration[11]	Exact integration[13,14]	Hybrid-Galerkin method, Eq. (12)
−3.0	0.29118	X[a]	4.8782	—	4.4834	4.5822
−2.1	0.28768	X	3.6541	—	3.2751	3.3482
−1.2	0.27612	X	2.4492	—	2.0690	2.1200
−0.6	0.24109	0.4069	1.6771	—	1.2609	1.3066
−0.3	0.15539	0.8896	1.3170	—	0.9327	0.9381
0.3	0.05945	0.8896	0.7802	0.9448	0.9448	0.9325
0.6	0.15239	0.4069	0.7489	0.9568	0.9568	0.9344
1.2	0.25155	X	1.2227	1.4376	1.4376	1.4373
2.1	0.28242	X	2.3391	2.5341	2.5342	2.5765
3.0	0.28933	X	3.5409	3.7137	3.7137	3.7883

[a]X = perturbation method fails.

the equation of motion (1) for this case are $\alpha = 0.064203$, $\beta = -0.72285$, and $\gamma = 4.0822$. The frequency ratios ω/ω_L for the specified values of the amplitude ratios ξ_{max} are presented in Table 1. The hybrid-Galerkin method, assuming only two terms in the approximate displacement function (6), yields good results that are comparable with those obtained by exact integration.[11,13,14] The accuracy of the results can be further improved by assuming more numbers of terms in the approximate displacement function.

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Fully Nonlinear Model of Cables

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Introduction

Applications of cables include power cables, information transmission lines, towing and mooring marine vehicles, lines for tethering objects over long distances, uses in large buildings to provide large, column-free areas, tension members of cable-stayed bridges or guy-towers, guy cables for wind turbines, taut strings for musical instruments, etc. Because engineering cables are usually lengthy and flexible, their vibrations are essentially nonlinear and dominated by geometric nonlinearities. Large-amplitude vibrations and nonlinear phenomena of cables have been extensively studied analytically[1–11] and experimentally.[3,6,7] There are many nonlinear cable models due to the many ad hoc assumptions adopted by different researchers during different steps in the derivation. Because extensional forces are the only elastic loads acting on cables, compressibility and the Poisson effect can be significant. For example, for rubber-like materials the Poisson effect results in a nonlinear relationship[12] between the tension force and the axial strain. In the literature, most of cable models do not account for the change in the cross-sectional area due to the Poisson effect or assume that the material is incompressible; hence, the Poisson ratio ν is absent from the equations of motion. Such theories are valid only for materials with $\nu \simeq 0$ or $\nu \simeq 0.5$. But, the Poisson ratios of most engineering materials are between 0.25 and 0.35, except those of rubber and paraffin, for which $\nu \simeq 0.5$. Most researchers account only for quadratic or cubic nonlinearities. However, quadratic and cubic nonlinearities have different influences on nonlinear responses of cables[1,11]; hence, both of them need to be modeled. For large-amplitude vibrations, nonlinear terms of order higher than cubic are needed. For example, to study cable vibrations with one to five internal resonances, fourth- and fifth-order terms are needed.[1] Hence, we present a geometrically exact or fully nonlinear cable model, which can be expanded to any order, taking into account initial sags, static loads, and extensionality.

In this Note we use an energy approach to develop a fully nonlinear cable model, which contains most cable models as special cases. The theory accounts for the effects of static and dynamic loads, initial sags, compressibility, material nonuniformity, Poisson's effect, and geometric nonlinearities. Also, we derive cubic nonlinear equations of motion for sagged cables, taut strings, and extensional bars and compare the model with several other models.

Fully Nonlinear Equations of Motion

Figure 1a shows the deformed configuration of a cable and the inertial coordinate system x_1-x_2-x_3. Point P_0 indicates the position of the observed particle when the cable is not loaded, point $\hat{P}$ is the deformed position of P_0 under static loads, and point P is the deformed position of P_0 under static and dynamic loads. The coordinates of $\hat{P}$ are (α_1, α_2, α_3); u_1, u_2, and u_3 are the dynamic displacements along the axes x_1, x_2, and x_3, respectively; and the coordinates of P are (x_1, x_2, x_3). Thus,

$$x_i = \alpha_i + u_i \qquad \text{for} \qquad i = 1, 2, 3 \tag{1}$$

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